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ON ACCESSIBLE GROUPS

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0. Introduction

In his important work on the ends of finitely generated groups, Stallings (see [1] or [6]) proved the following theorem

Structure Theorem. *Let G be a finitely generated group with more than one end, then either*

(α) *G is a free product with amalgamation,* $G = X *_K Y$ *with K finite and* $K \neq X$, $K \neq Y$, *or*

(β) *G is an* HNN-*group,* $G = \langle X, z \mid K^z = L \rangle$ *with K finite.*

The factors X (and Y) are easily seen to be finitely generated. Stalling's result is usually stated for groups with infinitely many ends, but a group with two ends satisfies either (a) with K of index two in X and Y or (β) with $K = L = X$ (see [6]). In (β) it is allowed that $X = \langle 1 \rangle$.

If X (or Y) has more than one end, it can itself be factorized, and so on. If this procedure stops, G is called accessible. Using the theory due to Bass and Serre (see [2]) of graphs of groups, an accessible group is precisely one which is the fundamental group of a finite graph of groups in which the edge groups are finite and the vertex groups have at most one end. C.T.C. Wall [7] conjectured that every finitely generated group is accessible. A proof of this conjecture would entail generalizing Gruschko's Theorem (see [1]) to free products with finite amalgamations.

Let M be a right $\mathbf{Z}G$-module and let Der(G,M) denote the set of all right derivations $d : G \to M$. These are the mappings which satisfy the rule

$$(xy)d = xdy + yd$$

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for all $x, y \in G$. Now if M is a two-sided $\mathbf{Z}G$-module, then $\mathrm{Der}(G, M)$ is a left G-module. Thus, for $g \in G$, $gd : G \to M$ is the mapping for which $y(gd) = g(yd)$. Let $D(G) = \mathrm{Der}(G, \mathbf{Z}G)$ and $A(G) = \mathbf{Z} \otimes_{\mathbf{Z}G} D(G)$ where $\mathbf{Z}$ is given the trivial structure as a right $\mathbf{Z}G$-module. The object of this paper is to investigate the structure of $A(G)$ and show that it contains information about whether G is accessible or not. Thus it will be proved that if G is accessible, then $A(G)$ is finitely generated, and that if G is almost finitely presented and $A(G)$ is finitely generated, then G is accessible.

Let $\mathrm{In}\, D(G)$ denote the set of inner derivations $d : G \to \mathbf{Z}G$. This is the submodule of $D(G)$ generated by $\epsilon_G : G \to \mathbf{Z}G$, $x\epsilon_G = 1 - x$, $x \in G$. Now

$$D(G)/\mathrm{In}\, D(G) \cong H^1(G, \mathbf{Z}G)$$

(see [3], §3.5) and $\mathbf{Z} \otimes \mathrm{In}\, D(G)$ is cyclic. Hence $A(G)$ is finitely generated if and only if $\mathbf{Z} \otimes_{\mathbf{Z}G} H^1(G, \mathbf{Z}G)$ is finitely generated. The structure of $H^1(G, \mathbf{Z}G)$ as a $\mathbf{Z}G$-module was originally discussed by Whitehead [8].

1. The abelian group $A(G)$

Throughout G will denote a finitely generated group.

Cohen ([1], p. 60) has demonstrated the relation between $D(G)$ and the almost invariant subsets of G, which is described here in some detail.

Let $\overline{\mathbf{Z}G}$ be the set of mappings from G to $\mathbf{Z}$. We regard $\overline{\mathbf{Z}G}$ as a two-sided $\mathbf{Z}G$-module by defining, for $\alpha, \beta \in \overline{\mathbf{Z}G}$, $x, g, h \in G$

$$x(\alpha + \beta) = x\alpha + x\beta\,, \quad x(g\alpha h) = (g^{-1}xh^{-1})\alpha\,.$$

(Mappings are written on the right.) Note that if α is written as an infinite formal sum $\alpha = \Sigma n_x x$ over the elements of G, where $n_x = x\alpha$, then $g\alpha h = \Sigma n_x gxh$.

If S is a subset of G, let $\chi_S : G \to \mathbf{Z}$ be its characteristic function, i.e. $x\chi_S = 1$ if $x \in S$, $x\chi_S = 0$ if $x \notin S$.

If $\alpha, \beta \in \overline{\mathbf{Z}G}$, we write $\alpha \overset{\mathrm{a}}{=} \beta$ if $x\alpha = x\beta$ for all but a finite number of $x \in G$.

Definition. An element α of $\overline{\mathbf{Z}G}$ is said to be almost invariant if $\alpha \overset{\mathrm{a}}{=} \alpha g$ for every $g \in G$. A subset S of G is said to be almost invariant if χ_S is almost invariant.

Lemma 1.1. *The set $M(G)$ of almost invariant elements of $\overline{\mathbf{Z}G}$ form a left submodule of $\overline{\mathbf{Z}G}$ generated by the characteristic functions of almost invariant subsets of G.*

There is an epimorphism $\theta : M(G) \to D(G)$ of left $\mathbf{Z}G$-modules, whose kernel is the set $C(G)$ of constant mappings.

Proof. Clearly $M(G)$ is a left submodule of $\overline{\mathbf{Z}G}$. Suppose $\alpha \in M(G)$. Let $E_i = \{x \mid x\alpha = i\}$. Now $\alpha \overset{\mathrm{a}}{=} \alpha g$ for every $g \in G$. Hence E_i is almost invariant. Also G is finitely generated, say $G = \langle x_1, x_2, ..., x_n \rangle$. Let

$$U = \{u \mid u\alpha \neq ux_j\alpha \text{ or } u\alpha \neq ux_j^{-1}\alpha \text{ for some } j, \text{ or } u = 1\}.$$

If $g \in G$, then g can be written as a word in the x_i's and their inverses. By considering the values of α on the initial segments of this word, it can be seen that $g\alpha = u\alpha$ for some $u \in U$. But U is finite, and so $E_i \neq \emptyset$ for all but a finite number of i and $M(G)$ is generated by the characteristic functions of the almost invariant subsets of G.

Let $\alpha \in M(G)$, then for each $g \in G$, $\alpha - \alpha g$ has finite support, and so can be regarded as an element of $\mathbf{Z}G$. Let

$$\theta : M(G) \to D(G)$$

$$x(\alpha\theta) = \alpha - \alpha x.$$

First we show that $\alpha\theta \in D(G)$. Let $x, y \in G$, then

$$(xy)\alpha\theta = \alpha - \alpha xy = \alpha(1 - x)y + \alpha(1 - y) = (x\alpha\theta)y + y(\alpha\theta).$$

Also θ is a homormophism of left $\mathbf{Z}G$-modules, since clearly θ is additive, and if $x, g \in G$, $\alpha \in M(G)$,

$$x(g(\alpha\theta)) = g(x\alpha\theta) = g(\alpha - \alpha x) = g\alpha - g\alpha\theta = x(g\alpha)\theta.$$

If $\alpha\theta$ is the zero mapping, then $\alpha = \alpha x$ for every $x \in G$, and so α is a constant mapping. Finally θ is surjective. For suppose $d \in D(G)$. Let $\alpha \in \overline{\mathbf{Z}G}$ be defined as follows:

$$x\alpha = \text{coefficient of } x \text{ in } xd,$$

then $\alpha \in M(G)$. For if $g \in G$,

$$\begin{aligned} x(\alpha g) = (xg^{-1})\alpha &= \text{coeff. of } xg^{-1} \text{ in } (xg^{-1})d \\ &= \text{coeff. of } xg^{-1} \text{ in } xdg^{-1} + g^{-1}d. \end{aligned}$$

But this is equal to the coefficient of x in xd unless x has non-zero coefficient in $g^{-1}dg$, which is true for only a finite number of elements $x \in G$. Also, for $g \in G$

$$\begin{aligned} g(\alpha - \alpha x) &= g\alpha - (gx^{-1})\alpha \\ &= \text{coeff. of } g \text{ in } gd - \text{coeff. of } gx^{-1} \text{ in } (gx^{-1})d \\ &= \text{coeff. of } gx^{-1} \text{ in } -x^{-1}d. \end{aligned}$$

Now $x^{-1}d = -xdx^{-1}$. Hence

$$g(\alpha - \alpha x) = \text{coeff. of } g \text{ in } xd.$$

Thus $\alpha - \alpha x = xd$, $\alpha\theta = d$ and so θ is surjective.

The structure of $D(G)$ and $A(G)$ in some simple cases can now be determined.

Lemma 1.2. *Let G be a finite group, $G = \{g_1, g_2, ..., g_k\}$. Put $w = g_1 + g_2 + ... + g_k$, then*

$$D(G) \cong \mathbf{Z}G/\mathbf{Z}Gw \quad \text{and} \quad A(G) \cong \mathbf{Z}_k ,$$

where $\mathbf{Z}_k$ denotes the cyclic group of order k.

Proof. Since G is finite, $M(G) = \overline{\mathbf{Z}G} = \mathbf{Z}G$ and $C(G) = \mathbf{Z}w = \mathbf{Z}Gw$. The result follows immediately.

Lemma 1.3. *If G has one end, then $D(G) \cong \mathbf{Z}G$ and $A(G) \cong \mathbf{Z}$.*

Proof. Since every almost invariant subset of G is finite or has finite complement (see [1]), it follows that $M(G)$ is generated by χ_G and $\chi_{\{1\}}$, the characteristic functions of G and $\{1\}$. Now $\chi_{\{1\}}$ generates a submodule of $M(G)$ isomorphic to $\mathbf{Z}(G)$ and χ_G generates $C(G)$. In fact it is easy to see that $M(G) \cong C(G) \oplus \mathbf{Z}G$ and the result follows from Lemma 1.1.

2. The functor A

Lemma 2.1. *Let H be a finitely generated subgroup of G. There is a canonical isomorphism of left $\mathbf{Z}G$-modules*

$$\nu : \mathrm{Der}(H, \mathbf{Z}G) \to \mathbf{Z}G \otimes_{\mathbf{Z}H} D(H) .$$

Proof. Choose a left transversal T for H in G. Let $d \in \mathrm{Der}(H, \mathbf{Z}G)$. For each $t \in T$ let $d_t : H \to \mathbf{Z}H$ be given by

$$hd = \sum_{t \in T} thd_t .$$

It is easy to check that d_t is a derivation for each $t \in T$. Also, a derivation is determined by its values on a generating set. Since H is finitely generated, it follows that $d_t = 0$ for almost all $t \in T$. Let

$$d\nu = \sum_{t \in T} t \otimes d_t .$$

It is easy to check that ν is bijective and independent of T. If $g \in G$, the set $T' = \{gt \mid t \in T\}$ is a left transversal of H in G. Hence $gd = \Sigma_{t' \in T'} t' \otimes (gd)_{t'}$. Now if $t' = gt$, $d_t = (gd)_{t'}$. Hence

$$(gd)\nu = \sum_{t \in T} gt \otimes d_t = g \sum_{t \in T} t \otimes d_t = g(d\nu) .$$

Let $\psi : D(G) \to \mathbf{Z}G \otimes D(H)$, $\psi = \rho\nu$, where $\rho : \mathrm{Der}(G, \mathbf{Z}G) \to \mathrm{Der}(H, \mathbf{Z}G)$ is the restriction mapping. It can be seen that ψ induces a homomorphism

$$\psi' : \mathbf{Z} \otimes D(G) \to \mathbf{Z} \otimes (\mathbf{Z}G \otimes D(H)) = \mathbf{Z} \otimes D(H) .$$

Thus, if $\iota : H \to G$ is the inclusion mapping, we have defined a homomorphism $A(\iota) = \psi' : A(G) \to A(H)$.

Theorem 2.2. *A is a contravariant functor from the category of finitely generated groups, in which morphisms are inclusion mappings, to the category of abelian groups.*

The proof is straightforward, if it is noted that if $K \leqslant H \leqslant G$, then the product of a left transversal for H in G and a left transversal for K in H is a left transversal for K in G.

Lemma 2.3. *Let H be a finitely generated subgroup of G and let $\iota : H \to G$ be the inclusion mapping. If H has at most one end, then $A(\iota) : A(G) \to A(H)$ is surjective.*

Proof. Let $\epsilon_G : G \to \mathbf{Z}G$ be the mapping for which $x\epsilon_G = 1 - x, x \in G$, then $\epsilon_G \in D(G)$. It is easy to see that $\epsilon_G \psi = 1 \otimes \epsilon_H$, and hence $(1 \otimes \epsilon_G)A(\iota) = 1 \otimes \epsilon_H$. But $h\epsilon_H = \chi_{\{1\}} - \chi_{\{1\}}h$. It follows from the proofs of Lemmas 1.2 and 1.3 that if H has at most one end, then $1 \otimes \epsilon_H$ generates $A(H)$. Hence $A(\iota)$ is surjective.

3. The structure of $A(G)$ when G has more than one end

First we state without proof an easy lemma on derivations.

Lemma 3.1. *Let G be a group with generating set S and with R as a set of defining relations (R is a set of words in $S \cup S^{-1}$). Let B be a right $\mathbf{Z}G$-module and let $d : S \to B$ be a mapping. Extend d to a mapping of the set of words in $S \cup S^{-1}$ by the formulae*

$$(s^{-1})d = -sds^{-1} , \quad s \in S$$

$$(wy)d = wdy + yd , \quad \text{where } w \text{ is a word in } S \cup S^{-1} \text{ and}$$

$$y \in S \cup S^{-1} .$$

If, for each $r \in R$, $rd = 0$, then d induces a derivation from G into B.

The structure theorem indicates that if G has more than one end, then there are two possibilities.

(α) $G = X *_K Y$, *where K is finite, $K \neq X$, $K \neq Y$.*

There is a pushout diagram

$$\begin{array}{ccc} K & \xrightarrow{i_1} & X \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ Y & \xrightarrow{j_2} & G \end{array}$$

of inclusion mappings. Note that X and Y are finitely generated (see [4], Theorem 4).

Theorem 3.2. *The diagram*

$$\begin{array}{ccc} A(G) & \longrightarrow & A(X) \\ \downarrow & & \downarrow \\ A(Y) & \longrightarrow & A(K) \end{array}$$

is a pullback diagram.

Proof. Consider the sequence

$$0 \to D(G) \xrightarrow{(\rho_X, \rho_Y)} \mathrm{Der}(X, \mathbf{Z}G) \oplus \mathrm{Der}(Y, \mathbf{Z}G) \xrightarrow{\binom{\mu_X}{-\mu_Y}} \mathrm{Der}(K, \mathbf{Z}G) \to 0$$

where $\rho_X, \rho_Y, \mu_X, \mu_Y$ are the appropriate restriction mappings. This sequence is exact. For, by Lemma 3.1, derivations $d_1 \in \mathrm{Der}(X, \mathbf{Z}G)$, $d_2 \in \mathrm{Der}(Y, \mathbf{Z}G)$ can be extended uniquely to G if and only if they are equal on K. Also $\mathrm{Der}(K, \mathbf{Z}G)$ $= \mathbf{Z}G \otimes D(K)$, is generated as a $\mathbf{Z}G$-module by ϵ_K. But

$$(\epsilon_X, 0)\begin{pmatrix} \mu_X \\ -\mu_Y \end{pmatrix} = \epsilon_K \;.$$

From the above exact sequence, we obtain, using Lemma 2.1, the exact sequence

$$\mathrm{Tor}_1(\mathbf{Z}, D) \to A(G) \to A(X) \oplus A(Y) \to A(K) \to 0 \;,$$

where $D = \mathrm{Der}(K, \mathbf{Z}G)$. Now, from Lemma 1.2,

$$D = \mathbf{Z}G \otimes D(K) \cong \mathbf{Z}G/\mathbf{Z}Gw \;,$$

where w is the sum of the elements in K. From the exact sequence

$$0 \to \mathbf{Z}Gw \to \mathbf{Z}G \to D \to 0 \;,$$

we obtain the exact sequence

$$\mathrm{Tor}_1(\mathbf{Z}, \mathbf{Z}G) \to \mathrm{Tor}_1(\mathbf{Z}, D) \to \mathbf{Z} \otimes \mathbf{Z}Gw \to \mathbf{Z} \otimes \mathbf{Z}G \to \mathbf{Z} \otimes D \to 0 \;,$$

i.e.

$$0 \to \mathrm{Tor}_1(\mathbf{Z}, D) \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_n \to 0$$

where n is the order of K. Thus $\mathrm{Tor}_1(\mathbf{Z}, D) = 0$ and the theorem is proved.

Corollary 3.3. *The mappings $A(j_1)$, $A(j_2)$ are surjective. If $A(j_1)$ is an isomorphism, then $A(i_2)$ is an isomorphism.*

Proof. By Lemma 2.3, $A(i_1)$ and $A(i_2)$ are surjective. The result follows immediately from Theorem 3.2.

(β) *$G = \langle X, z; K^z = L\rangle$, K finite.*

It is not difficult to prove that, since G is finitely generated, X must be also.

Let $d \in \operatorname{Der}(X, \mathbf{Z}G)$. Define

$$d' : K \to \mathbf{Z}G$$

$$kd' = (z^{-1}kz)dz^{-1} .$$

Then $d' \in \operatorname{Der}(K, \mathbf{Z}G)$. For

$$(z^{-1}k_1k_2z)dz^{-1} = (z^{-1}k_1z)dz^{-1}k_2 + (z^{-1}k_2z)dz^{-1}$$

for every $k_1, k_2 \in K$. Define

$$\eta_X : \operatorname{Der}(X, \mathbf{Z}G) \to \operatorname{Der}(K, \mathbf{Z}G)$$

$$d\eta_X = d|_K - d' .$$

Clearly η_X is a homomorphism of left $\mathbf{Z}G$-modules. Let $\eta_1 : \mathbf{Z}G \to \operatorname{Der}(K, \mathbf{Z}G)$ be defined by $u\eta_1 = u\epsilon_K$ for $u \in \mathbf{Z}G$. Let $\rho_X : D(G) \to \operatorname{Der}(X, \mathbf{Z}G)$ be the restriction mapping, and let $\rho_1 : D(G) \to \mathbf{Z}G$ be given by

$$d\rho_1 = -zdz^{-1}$$

for $d \in D(G)$. Now for $k \in K$

$$(z^{-1}kz)d = -zdz^{-1}kz + kdz + zd .$$

Hence

$$kd - (z^{-1}kz)dz^{-1} = -zdz^{-1}(1-k) ,$$

i.e. $k(d\rho_X\eta_X) = k(d\rho_1\eta_1)$. Thus the diagram

$$\begin{tikzcd} D(G) \arrow[r, "\rho_1"] \arrow[d, "\rho_X"'] & \mathbf{Z}G \arrow[d, "\eta_1"] \\ \operatorname{Der}(X, \mathbf{Z}G) \arrow[r, "\eta_X"] & \operatorname{Der}(K, \mathbf{Z}G) \end{tikzcd}$$

is commutative.

Theorem 3.4. *The diagram*

$$\begin{array}{ccc} A(G) & \xrightarrow{1\otimes\rho_1} & \mathbf{Z} \\ {\scriptstyle 1\otimes\rho_X}\downarrow & & \downarrow{\scriptstyle 1\otimes\eta_1} \\ A(X) & \xrightarrow{1\otimes\eta_X} & A(K) \end{array}$$

is a pullback diagram. (Note that although $1\otimes\rho_X = A(j_X)$, where $j_X : X\to G$ is the inclusion mapping, $1\otimes\eta_X$ is not necessarily equal to $A(i_K)$ where i_K is the inclusion mapping of K in X.)

Proof. Consider the sequence

$$0\to D(G)\xrightarrow{(\rho_1,\rho_X)} \mathbf{Z}G\oplus \mathrm{Der}(X,\mathbf{Z}G)\xrightarrow{\binom{\eta_1}{-\eta_X}} \mathrm{Der}(K,\mathbf{Z}G)\to 0\,.$$

Since $\rho_1\eta_1 - \rho_X\eta_X = 0$, the product of each pair of consecutive mappings is zero. Also, if $d\in \mathrm{Der}(X,\mathbf{Z}G)$ and $u\in\mathbf{Z}G$, then it follows from Lemma 3.1 that d can be extended uniquely to a derivation $d_1\in D(G)$ for which $-zd_1z^{-1}=u$ if and only if $kd-(z^{-1}kz)dz^{-1}=u(1-k)$ for each $k\in K$, i.e. if and only if $u\eta_1 - d\eta_X = 0$. It follows that the above sequence is exact, since clearly η_1 is surjective.

As in the proof of Theorem 3.3, it follows that there is an exact sequence

$$\mathrm{Tor}_1(\mathbf{Z},D)\to A(G)\to A(X)\oplus\mathbf{Z}\to A(K)\to 0\,.$$

But it has been shown that $\mathrm{Tor}_1(\mathbf{Z},D)=0$, and the theorem is proved.

Corollary 3.5. *If $j_X : X\to G$ is the inclusion mapping, then $A(j_X): A(G)\to A(X)$ is surjective with non-trivial kernal.*

Proof. This follows because $1\otimes\eta_1 : \mathbf{Z}\to A(K)$ is surjective with non-trivial kernel.

With the aid of Theorems 3.2 and 4.3 it is possible to compute $A(G)$ for any accessible group G.

Theorem 3.6. *If G is accessible, then $A(G)$ is finitely generated.*

Proof. Suppose G is the fundamental group of a finite graph of groups, in which the edge groups are finite and the vertex groups $G_1, G_2, \dots, G_n$ each have at most one end. Suppose the graph has e edges and v vertices, and put $p=e-v+1$. By Theorems 3.4 and 3.6 it follows that $A(G)$ is isomorphic to a subgroup of $A(G_1)\oplus A(G_2)\oplus\dots\oplus A(G_n)\oplus\mathbf{Z}^p$, where $\mathbf{Z}^p$ denotes the direct sum of p copies of $\mathbf{Z}$. But by Lemmas 1.2 and 1.3, $A(G_i)$ is cyclic for $i=1,2,\dots,n$. Hence $A(G)$ is finitely generated.

Definition. A subgroup H of G is called a factor of G, if there is a finite sequence of subgroups of G,

$$G = X_0 > X_1 > \ldots > X_n = H$$

where either $X_{i-1} = X_i *_{K_i} Y_i$, K_i finite, $K_i \neq X_i$, $K_i \neq Y_i$, or $X_{i-1} = \langle X_i, z_i | z_i^{-1} K_i z_i = L_i \rangle$, K_i finite, $i = 1, 2, \ldots, n$.

Theorem 3.7. *If $A(G)$ is finitely generated, then either*
(i) *G is accessible,*
or (ii) *G has a finite subgroup F contained in an infinite factor H, and $A(\iota): A(H) \to A(F)$ is an isomorphism.*

Proof. If G has at most one end then G is accessible. Suppose G has more than one end. Thus G has one of the decompositions described in the structure theorem. If $G = X *_K Y$ with K finite, $K \neq X$, $K \neq Y$, and condition (ii) is not satisfied, then, by Theorem 3.2, the mappings $A(G) \to A(X)$, $A(G) \to A(Y)$ are surjective with non-trivial kernels. Similarly if $G = \langle X, z; K^z = L \rangle$ with K finite, then, by Theorem 3.4, the mapping $A(G) \to A(X)$ is surjective with non-trivial kernel. Since $A(G)$ is a finitely generated abelian group, there can be no infinite sequence

$$A(G) \to A(X_1) \to A(X_2) \to \ldots$$

where each mapping is surjective with non-trivial kernel. Hence the process of successively decomposing the factors of G must terminate, and so G is accessible.

4. Almost finitely presented groups

Following Stallings [5], call a group G almost finitely presented if there is some finite simplicial complex K with regular covering space $\tilde{K}$, having G as its group of covering translations and such that $H^1(\tilde{K}) = 0$ with $\mathbf{Z}_2$ coefficients. Thus $\tilde{K}$ is connected, locally finite and G acts freely on $\tilde{K}$. It may as well be assumed that K is two-dimensional.

Every finitely presented group is almost finitely presented [5, (3.3)].

Using $\mathbf{Z}_2$ coefficients, an n-cochain of $\tilde{K}$ can be regarded as a set of n-simplexes. Let E be a 0-cochain. Let E^* denote its complement, then δE, the coboundary of E, consists of those 1-simplexes which have one vertex in E and one vertex in E^*. The set E is said to be connected if each pair of vertices of E can be joined by a path consisting of 1-simplexes none of which lies in δE.

Lemma 4.1. *If E is a 0-cochain of $\tilde{K}$ with E and E^* connected, then δE is connected.*

Proof. Let L be the subcomplex of $\tilde{K}$ consisting of all the 2-simplexes which have at least two vertices in E, all the 1-simplexes which have at least one vertex in E, and all the vertices of these simplexes. Let L^* be defined similarly for E^*, then $L \cup L^* = \tilde{K}$, $L \cap L^* = \delta E$ and, since E and E^* are connected, so are L and L^*. Thus there is an exact Mayer-Vietoris sequence in reduced cohomology with $\mathbf{Z}_2$-coefficients:

$$0 \to \tilde{H}^0(\tilde{K}) \to \tilde{H}^0(L) \oplus \tilde{H}^0(L^*) \to \tilde{H}^0(\delta E) \to \tilde{H}^1(\tilde{K}) \to \dots .$$

But $\tilde{H}^0(L) = \tilde{H}^0(L^*) = 0$, since L and L^* are connected, and $\tilde{H}^1(\tilde{K}) = H^1(\tilde{K}) = 0$. Hence $\tilde{H}^0(\delta E) = 0$ and so δE is connected.

If $g \in G$ and σ is a simplex of $\tilde{K}$, then σ transformed by g is written $g\sigma$. Let F be a finite subset of $\tilde{K}$ which contains precisely one simplex above each simplex of K. If E is a 0-cochain of $\tilde{K}$ with finite coboundary δE, let $b(E)$ denote the number of elements of δE. Put $b(E) = 0$, if $E = \emptyset$.

Lemma 4.2. *Let E be a 0-cochain, and suppose $P = \delta E$ is finite, non-empty and connected, then, for every $g \in G$, at least one of $b(E \cap gE)$, $b(E \cap gE^*)$, $b(E^* \cap gE)$, $b(E^* \cap gE^*)$ is less than $b(E)$.*

Proof. Suppose each of $E \cap gE$, $E \cap gE^*$, $E^* \cap gE$, $E^* \cap gE^*$ is non-empty, then $P \cap gP \neq \emptyset$. For P is connected, and it has vertices in both gE and gE^*. Thus $P \cup gP$ has less than $2b(E)$ elements. Now each 1-simplex of $P \cup gP$ is contained in precisely two of the sets $\delta(E \cap gE)$, $\delta(E \cap gE^*)$, $\delta(E^* \cap gE)$, $\delta(E^* \cap gE^*)$. Therefore

$$b(E \cap gE) + b(E \cap gE^*) + b(E^* \cap gE) + b(E^* \cap gE^*) < 4b(E) ,$$

and the lemma follows immediately.

Since $\tilde{K}$ is locally finite, there are only a finite number of connected sets with a fixed number of 1-simplexes which intersect F. It follows that there exists an infinite sequence $P_1, P_2, \dots$ of non-empty, finite, connected 1-cocycles of $\tilde{K}$ (each P_i is the coboundary of two 0-cochains E_i and E_i^*) which satisfies the following conditions:

(a) if $gP_i = P_j$ for some $g \in G$, then $i = j$ and $g = 1$,

(b) if P is a connected, finite 1-cocycle, then $P = gP_i$ for some $i = 1, 2, \dots$, and some $g \in G$,

(c) $b(E_i) \leqslant b(E_j)$ if $i < j$.

It follows from Lemma 4.2 that for every $i = 1, 2, \dots$, and every $g \in G$, at least one of $b(E_i \cap gE_i)$, $b(E_i \cap gE_i^*)$, $b(E_i^* \cap gE_i)$, $b(E_i^* \cap gE_i^*)$ is less than $b(E_i)$.

If E is a 0-cochain of $\tilde{K}$ with δE finite, and v_0 is a fixed vertex of F, then

$$\tau E = \{g \mid gv_0 \in E\}$$

is an almost invariant subset of G. To see this, note that any path of 1-simplexes connecting vertices v_1 and v_2 of $\tilde{K}$, transforms under g to a path connecting gv_1 and gv_2, and if E has finite coboundary, this path will intersect δE for at most a finite number of elements $g \in G$. Hence for almost all $g \in G$, gv_1 and gv_2 both lie in

E or both lie in E^*. Thus

$$\tau E = \{g \mid gv_0 \in E\} \stackrel{a}{=} \{g \mid ghv_0 \in E\} = \tau E h^{-1} ,$$

for every $h \in G$. Conversely, if S is an almost invariant subset of G, let

$$\kappa S = \{sv \mid s \in S, v \text{ is a vertex of } F\} .$$

Let $\sigma \in \delta F$, then $\sigma = (v, gv')$ for some vertices v, v' of F and $g \in G$. Let $U = \{g \mid (v, gv') \in \delta F\}$, then U is finite since $\widetilde{K}$ is locally finite. Let

$$T = \{s \mid s \in S, su \notin S \text{ for some } u \in U\} .$$

Now T is finite since S is almost invariant. But each element of $\delta\kappa S$ must have a vertex of the form tv, where $t \in T$ and $v \in F$. Hence $\delta\kappa S$ is finite. Clearly $\tau\kappa S = S$.

Put $e = \chi_G$ and $m_i = \chi_{\tau E_i}$, $i = 1, 2, ...$, where the $P_i = \delta E_i$ satisfy conditions (a), (b) and (c). Let M_0 be the submodule of $M(G)$ generated by e, and for $k > 0$, let M_k be the submodule generated by $e, m_1, m_2, ..., m_k$.

Lemma 4.3. *If E is a 0-cochain of $\widetilde{K}$ with $b(E) < b(E_k)$, then $\chi_{\tau E} \in M_{k-1}$.*

Proof. The proof is by induction on $b(E)$. Clearly the result is true if δE is connected. If δE is not connected, then it follows from Lemma 4.1 that E or E^* is not connected. If, say, E is not connected, let C be a connected component of E. We have $E = C \cup D$, $D \neq \emptyset$, $C \cap D = \emptyset$, $\delta E = \delta C \cup \delta D$ and $\delta C \cap \delta D = \emptyset$. By the induction hypothesis $\chi_{\tau C}, \chi_{\tau D} \in M_{k-1}$. Hence $\chi_{\tau E} = \chi_{\tau C} + \chi_{\tau D} \in M_{k-1}$, and the lemma is proved.

Let $\theta : M(G) \to D(G)$ be the homomorphism defined in Lemma 1.1. Put $d_i = m_i\theta$, $i = 1, 2, ...$.

Lemma 4.4. *The derivations d_i, $i = 1, 2, ...,$ generate $D(G)$.*

Proof. By definition, $M(G)$ is generated by the characteristic functions of the almost invariant subsets S of G. But $S = \tau\kappa S$. It follows from Lemma 4.3 that $M(G)$ is generated by the set $e, m_1, m_2, ...$. Since $e\theta = 0$, it follows that $D(G)$ is generated by $d_1, d_2, ...$.

Recall that $\epsilon_G : G \to \mathbf{Z}G$ is the mapping $g\epsilon_G = 1 - g$.

Lemma 4.5. *Let S be an almost invariant subset of G and let $d = \chi_S\theta$. Let H be a finitely generated subgroup of G and let $\rho_H : D(G) \to \mathrm{Der}(H, \mathbf{Z}G)$ be the restriction mapping. Then $d\rho_H = f\epsilon_G\rho_H$ for some $f \in \mathbf{Z}G$ if and only if S is almost equal to a union of left cosets of H.*

Proof. If $S \stackrel{a}{=} U$, where U is a union of left cosets of H, then $\chi_S - \chi_U = f$ can be regarded as an element of $\mathbf{Z}G$. Also $\chi_U h = \chi_U$ for every $h \in H$. Hence $\chi_U\theta\rho_H = 0$, and so $d\rho_H = f\theta\rho_H = f\epsilon_G\rho_H$. Conversely, if $\chi_U\theta\rho_H = f\epsilon_G\rho_H$, there is a finite set F such that $\chi_U\theta\rho_H = \chi_F\theta\rho_H$ modulo 2. If $U \triangle F$ is the disjoint union of U and F, it is easy to see that $(U \triangle F)h = U \triangle F$ for all $h \in H$ and the lemma follows at once.

Note that it follows from [1], Lemma 2.3, that for almost all cosets gH of H either $gH \subseteq S$ or $gH \subseteq G - S = S^*$.

Lemma 4.6. *Let S be an almost invariant subset of G. Suppose S and S^* are infinite, but that for every $g \in G$ at least one of $S \cap gS$, $S \cap gS^*$, $S^* \cap gS$, $S^* \cap gS^*$ is finite, then G can be decomposed*

$$G = X *_K Y\,, \quad K \text{ finite},\ K \neq X,\ K \neq Y\,,$$

or

$$G = \langle X, z \mid z^{-1}Kz = L \rangle\,, \quad K \text{ finite},$$

in such a way that S is almost equal to a union of left cosets of X (or Y).

Proof. This follows from the proof of the structure theorem in [1]. For put $K = \{g \mid gS \overset{a}{=} S\}$, $H = \{g \mid gS \overset{a}{=} S$ or $gS \overset{a}{=} S\}$ and $S_1 = \{g \mid S^* \cap gS$ or $S^* \cap gS^*$ finite$\}$, then it is shown in [1], that $S_1 \overset{a}{=} S$, $K \leqslant H$ are finite subgroups of G, and G has a decomposition as given above in which

$$X = \{g \mid (S_1 - K)g = S_1 - K\}$$

and

$$Y = H \text{ or } \{g \mid (S_1 - (H - K))g = S_1 - (H - K)\}\,.$$

If $Y = H$, then, since H is finite and almost all cosets of H lie in S or S^* it follows that S is almost equal to a union of left cosets of Y. The result follows in the other cases because $S \overset{a}{=} S_1 \overset{a}{=} S_1 - K \overset{a}{=} S_1 - (H - K)$.

Definition. A finitely generated group G is said to be β-indecomposable if it has no decomposition as an HNN-group $G = \langle X, z \mid K^z = L\rangle$ with K finite.

From the theory of graphs of groups, it can be shown that if G is β-indecomposable then so is every factor of G. If $A(G)$ is finite, then G is β-indecomposable by Theorem 3.4.

Lemma 4.7. *Let G be a β-indecomposable group. Let $S_1, S_2, \ldots, S_k$ be almost invariant subsets of G each of which satisfy the conditions of Lemma 4.6, then G can be decomposed $G = X *_K Y$ with K finite, $K \neq X$, $K \neq Y$, in such a way that each S_i is almost equal to a union of left cosets of X (and Y).*

Proof. The proof is by induction on k. The result for $k = 1$ follows from Lemma 4.6. Assume the result is true for $k - 1$. Thus

$$G = A *_C B\,, \quad C \text{ finite}, \quad C \neq A\,, \quad C \neq B\,,$$

where each of $S_1, S_2, \ldots, S_{k-1}$ is almost equal to a union of left cosets of A. Again, by Lemma 4.6, G has a decomposition $G = X' *_{K'} Y'$, where S_k is almost equal to a union of left cosets of X' and of Y'. Now A is β-indecomposable since G is β-indecomposable. By Theorem 6 of [4], A can be decomposed as a tree product of factors each of which is the intersection of A with a conjugate of X' or Y', and for which edge groups are finite. By Theorem 4 of [4] the tree Γ can be taken to be finite, and we may assume that edge maps are not surjective. It can also be assumed that C lies in one of the vertex groups. For C is finite, and so some conjugate of C must lie in one of the vertex groups. The decomposition of A can be adjusted to ensure that this conjugate is C itself. Let v_0 denote a vertex of Γ for which C is contained in the corresponding vertex group. There are at least two vertices of Γ which belong to just one edge. Thus there is such a vertex v for which $v \neq v_0$. Let X be the corresponding vertex group. It can be shown that $G = X *_K Y$, K finite, $K \neq X$, $K \neq Y$.

Now X is a subgroup of A, so each S_i, $i = 1, \ldots, k-1$, is almost equal to a union of left cosets of X. Also X lies in a conjugate of X' or Y', say $X \subseteq g^{-1}X'g$. Since S_k is almost equal to a union of left cosets of X', we find $S_k g$ (and hence the almost equal set S_k) almost equals a union of left cosets of $g^{-1}X'g$. The lemma follows immediately.

Lemma 4.8. *If D_1 is a finite subset of $D(G)$ and H is a finitely generated β-indecomposable subgroup of G, then for some factor Q of H, $d\rho_Q = q(d)\epsilon_Q$, where $q(d) \in \mathbf{Z}G$ for each $d \in D_1$.*

Proof. By Lemma 4.4, we can take D_1 to be the set $\{d_1, d_2, \ldots, d_r\}$ for some $r > 0$, where $d_i = \chi_{\tau E_i}\theta$. Let s be the smallest integer for which $d_s\rho_H$ is not a left multiple of ϵ_H (or more correctly $\epsilon_G\rho_H$). If no such s exists, the lemma is trivially true. By Lemma 4.5 and the remark following it, there is at least one left coset gH for which $gH \cap \tau E_s$ and $gH \cap \tau E_s^*$ are both infinite. Let $g_1H, g_2H, \ldots, g_kH$ be the set of all left cosets with this property. This set is finite by the remark following Lemma 4.5. Let $S_i = H \cap g_i^{-1}\tau E_s$, $i = 1, 2, \ldots, k$. Let $S = H \cap g^{-1}\tau E_s$, be one of these sets, and put $S^* = H - S$. Then for every $h \in H$ at least one of $S \cap hS$, $S \cap hS^*$, $S^* \cap hS$, $S^* \cap hS^*$ is finite. For suppose, say $b(g^{-1}E_s^* \cap hg^{-1}E_s) < b(E_s)$, then putting $W = g^{-1}E_s^* \cap hg^{-1}E_s$, we see that $\chi_{\tau W}\theta$ lies in the submodule of $D(G)$ generated by $d_1, d_2, \ldots, d_{s-1}$. Hence $\chi_{\tau W}\theta\rho_H = f\epsilon_H$ for some $f \in \mathbf{Z}G$. Therefore τW is almost equal to a union of left cosets of H. But $H \cap \tau W = S^* \cap hS$ has infinite complement, and so $S^* \cap hS$ is finite. By Lemma 4.7, H can be decomposed $H = X *_K Y$, K finite, $K \neq X$, $K \neq Y$, in such a way that every S_i is almost equal to a union of left cosets of X. Also, then τE_s is a union of left cosets of X, and so, by Lemma 4.5, $d_s\rho_X = f\epsilon_X$ for some $f \in \mathbf{Z}G$. Repeat the process with X replacing H and so on. Eventually a factor Q is obtained satisfying the required condition.

Now we can prove the main result of this section.

Theorem 4.9. *If G is an almost finitely presented group and $A(G)$ is finitely generated, then G is accessible.*

Proof. Suppose G has an infinite factor H containing a finite subgroup F such that $A(\iota): A(H) \to A(F)$ is an isomorphism. Suppose F has smallest order n consistent with these conditions. Now $1 \otimes n\epsilon_H = 0$, since $A(F) \cong \mathbf{Z}_n$. Hence $n\epsilon_H = f_1 h_1 + f_2 h_2 + \ldots + f_r h_r$, for some $h_i \in D(H), f_i \in IH$, the augmentation ideal of $\mathbf{Z}H$, $i = 1, 2, \ldots, r$. By the proofs of Theorems 3.2 and 3.4, there exist $g_i \in D(G)$, for which $g_i \rho_H = h_i$, $i = 1, 2, \ldots, r$. By Lemma 4.8, H has a factor Q for which $g_i \rho_Q = q_i \epsilon_Q$, for $q_i \in \mathbf{Z}G$, $i = 1, 2, \ldots, r$. Hence

$$n\epsilon_Q = (f_1 q_1 + \ldots + f_r q_r)\epsilon_Q \,.$$

If Q is infinite, then $f\epsilon_Q = 0$ implies $f = 0$. But $n \notin IG$, and so there is a contradiction. Thus H has a finite factor Q.

Since H is β-indecomposable, it follows that $H = X *_K Y$, K finite, $K \neq X$, $K \neq Y$. Suppose $|K| < n$. By Corollary 3.3 either $|A(X)| < n$ or $A(Y) \to A(K)$ is an isomorphism. If $A(Y) \to A(K)$ is an isomorphism, then Y is infinite. But this contradicts the minimality of n. Thus $A(X) < n$. Similarly $A(Y) < n$. But either X or Y contains a conjugate of F, contradicting Lemma 2.3. Thus $|K| = n$ and X and Y are both infinite. We see that X and Y both satisfy the same conditions as H, and so successively decomposing X and Y can only give infinite factors. But H has a finite factor Q, and so we have a contradiction.

Thus no infinite factor H of G contains a finite subgroup F for which $A(H) \cong A(F)$, and the theorem follows from Theorem 3.7.

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